Robotics Systems Assignment 2

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1 Introduction

This report aims to demonstrate and confirm the viability of the robot arm design for the required task. Firstly, a Jacobian matrix is derived, which maps the joint velocities to the end-effector's velocity within a given frame of coordinates. Secondly, the Jacobian matrices that relate the joint velocities to the centres of mass of the links are derived. Lastly, the required joint torques to lift the inertia of the robot in addition to the load within the required workspace are calculated and checked against the nominal torque of the provided motors.

2 Kinematics and Jacobian Derivation

2.1 Robot Kinematics Overview

As a continuation and a quick recap of Assignment 1, the robot arm has 4 DOF, as illustrated in the figure 1.

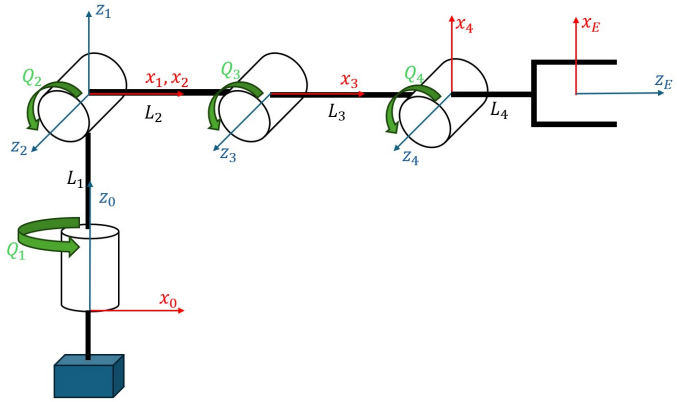


Figure 1: Robot arm configuration

With respect to the frames defined in figure 1, the transformation matrices between each respective frames are as follows:

$$
{}^{0}_{1}T = \begin{bmatrix} \cos(Q_1) & -\sin(Q_1) & 0 & 0 \\ \sin(Q_1) & \cos(Q_1) & 0 & 0 \\ 0 & 0 & 1 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^{1}_{2}T = \begin{bmatrix} \cos(Q_2) & -\sin(Q_2) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin(Q_2) & \cos(Q_2) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^{2}_{3}T = \begin{bmatrix} \cos(Q_3) & -\sin(Q_3) & 0 & L_2 \\ \sin(Q_3) & \cos(Q_3) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^{3}_{4}T = \begin{bmatrix} \cos(Q_4 + \frac{\pi}{2}) & -\sin(Q_4 + \frac{\pi}{2}) & 0 & L_3 \\ \sin(Q_4 + \frac{\pi}{2}) & \cos(Q_4 + \frac{\pi}{2}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^{4}_{E}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -L_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}
$$

Finally, to obtain the forward kinematics of ${}^{0}_{E}T$, the transformation matrices can be chain multiplied:

$$
{}^{0}_{E}T = {}^{0}_{1}T\,{}^{1}_{2}T \cdots {}^{n-1}_{n}T\,{}^{n}_{E}T \tag{2}
$$

2.2 Jacobian Matrix Derivation

The Jacobian matrix $J(q) \in R^{m \times n}$ describes the relationship between joint velocity and task space velocity at a given point of the robot, which is the end-effector in this case. This relation is described by equation 3 as follows:

$$
\begin{bmatrix} v_E \\ \omega_E \end{bmatrix} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \begin{bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{Q}_3 \\ \dot{Q}_4 \end{bmatrix} \tag{3}
$$

The terms v_E and ω_E represents the translational velocity and angular velocity of the end effector E. J_v is the Jacobian for translational velocity, and J_ω is the Jacobian for angular angular velocity components. The overall Jacobian J is given by equation 4:

$$
J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \tag{4}
$$

To make the derivation of the Jacobian matrix simpler, J_v and J_ω can first be separately derived, then combining them at the end.

2.2.1 Deriving J_v

The columns of J_v are given by ${}^0\hat{z}_i \times {}^0P_{iE}$ for each revolute joint i, represented in frame 0. Each transformation matrix can be broken down into the form indicated in equation 5, where ${}^a_bR$ represents the rotation matrix that rotates a vector in frame b onto frame a. ${}^aP_{ab}$ is the displacement from a to b in frame a.

$$ {}^a_bT = \left[\begin{array}{ccc|c} & {}^a_bR & & {}^aP_{ab} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{5} $$

Since we already have access to all of the transformation matrices, the ${}^0\hat{z}_i$ is already contained within each rotation matrix ${}^0_iR$ in ${}^0_iT$ as the third column, which is the unit vector of z for frame i represented in frame 0. For example:

$$ {}^0_2T = \begin{bmatrix} \cos(Q_1)\cos(Q_2) & -\cos(Q_1)\sin(Q_2) & \sin(Q_1) & 0 \\ \cos(Q_2)\sin(Q_1) & -\sin(Q_1)\sin(Q_2) & -\cos(Q_1) & 0 \\ \sin(Q_2) & \cos(Q_2) & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, {}^0\hat{z}_2 = \begin{bmatrix} \sin(Q_1) \\ -\cos(Q_1) \\ 0 \end{bmatrix} $$

${}^0P_{iE}$ can be found by taking the top right displacement vector of ${}^i_ET$ matrix, which is ${}^iP_{iE}$. Although this is the correct vector, it is still in the wrong frame. It needs to be rotated from frame i to frame 0. Hence:

$$ {}^0P_{iE} = {}^0_iR \times {}^iP_{iE} \tag{6} $$

Now to put it all together to find J_v:

$$ J_v = \begin{bmatrix} {}^0\hat{z}_1 \times {}^0P_{1E} & {}^0\hat{z}_2 \times {}^0P_{2E} & {}^0\hat{z}_3 \times {}^0P_{3E} & {}^0\hat{z}_4 \times {}^0P_{4E} \end{bmatrix} \tag{7} $$

2.2.2 Deriving J_ω

Deriving J_ω is simpler than J_c, as each of the columns are ${}^0\hat{z}_i$, which has been found already in section 2.2.1.

$$ J_\omega = \begin{bmatrix} {}^0\hat{z}_1 & {}^0\hat{z}_2 & {}^0\hat{z}_3 & {}^0\hat{z}_4 \end{bmatrix} \tag{8} $$

2.3 Jacobian Verification

Referring to equation 4, J_v and J_ω can be 'stacked' to obtain J. All expressions to this point are expressed symbolically to keep the Jacobian simple. The exact Jacobian value in terms of joint angles is included in the MATLAB file. The Jacobian is a 6×4 matrix, with corresponding output of 3 linear and 3 angular velocities, as well as the input of the 4 joint velocities. To verify the validity of the Jacobian, it can be given some joint velocities, and see if it outputs the expected joint linear velocity and angular velocities.

2.3.1 Scenario 1

$$L = \begin{bmatrix} 0.15 \\ 0.25 \\ 0.2 \\ 0.12 \end{bmatrix}, Q = \begin{bmatrix} 0 \\ \frac{\pi}{4} \\ -\frac{\pi}{4} \\ \frac{\pi}{4} \end{bmatrix}, \dot{Q} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} v_E \\ \omega_E \end{bmatrix} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \cdot \dot{Q} = \begin{bmatrix} 0 \\ \frac{37\sqrt{2}}{200} + \frac{1}{5} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This value is to be expected, as $Q_1 = 0$, so the arm starts on the X-Z plane. Combining this with the only velocity component being $\dot{Q}_1 = 1$, the answer should exhibit an instantaneous linear velocity only in the y component. The $1\ \mathrm{rad\,s^{-1}}$ component is also to be expected, as a result of $\dot{Q}_1 = 1$.

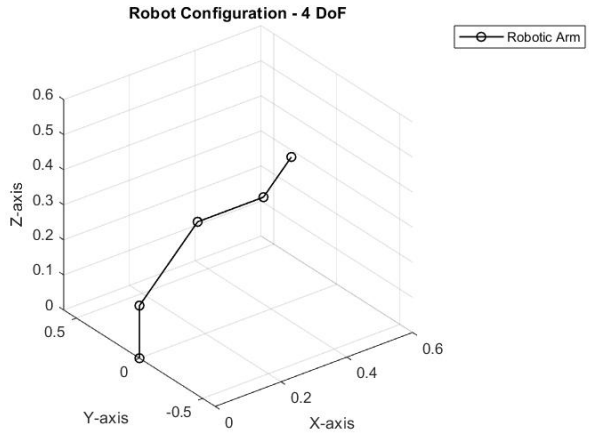


Figure 2: Initial position of scenario 1

2.3.2 Scenario 2

$$L = \begin{bmatrix} 0.15 \\ 0.25 \\ 0.2 \\ 0.12 \end{bmatrix}, Q = \begin{bmatrix} \frac{\pi}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \dot{Q} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} v_E \\ \omega_E \end{bmatrix} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \cdot \dot{Q} = \begin{bmatrix} 0 \\ 0 \\ \frac{57}{100} \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The arm now has an initial starting position that is indicated in figure 3, where it starts on the Y-Z plane. $\dot{Q}_2 = 1$ results in linkages 2, 3 and 4 all tilting upwards. This causes an instantaneous velocity in the positive z-direction, which is consistent with the results. Similarly, the 1 $\mathrm{rad\,s^{-1}}$ component is a result of $\dot{Q}_2 = 1$.

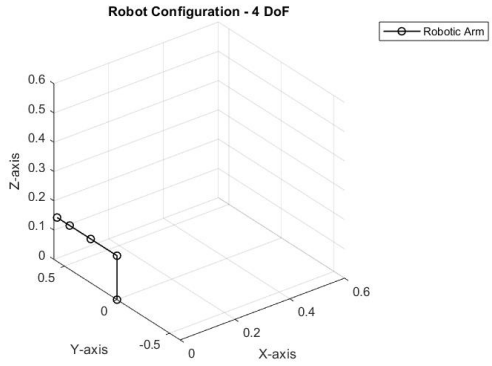


Figure 3: Initial position of scenario 2

2.3.3 Scenario 3

$$L = \begin{bmatrix} 0.15 \\ 0.25 \\ 0.2 \\ 0.12 \end{bmatrix}, Q = \begin{bmatrix} \frac{\pi}{4} \\ \frac{\pi}{4} \\ -\frac{\pi}{4} \\ \frac{\pi}{4} \end{bmatrix}, \dot{Q} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} v_E \\ \omega_E \end{bmatrix} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \cdot \dot{Q} = \begin{bmatrix} -\frac{\sqrt{2}\left(\frac{37\sqrt{2}}{200}+\frac{1}{5}\right)}{2} \\ \frac{\sqrt{2}\left(\frac{37\sqrt{2}}{200}+\frac{1}{5}\right)}{2} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The arm now has a initial starting position that is indicated in figure 4. The result indicates that it has velocity components in both x and y directions

and nothing in z, which is to be expected as the end-effector is not moving vertically, only horizontally in circular motion. Since the starting point is $Q_1 = \frac{\pi}{4}$, and the end-effector is undergoing circular motion, it makes sense for the magnitudes for the x and y components of the linear velocity to have the same magnitude.

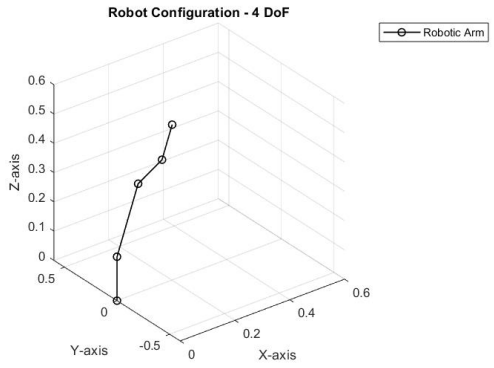


Figure 4: Initial position of scenario 3

For the 3 scenarios where the starting condition, initial joint angles as well as joint velocities were used to validate the Jacobian matrix, and the results were consistent with what is expected. These are good indicators that the Jacobian is correct.

3 Joint Torque Calculations

3.1 Jacobian Matrices of each Centre of Mass

The process of the subsequent derivation of the Jacobian matrices is as follows. *Note: The detailed symbolic representations of rotation matrices and unit vectors can be found in Appendix A. The subsequent position vectors have been defined and can be found in Appendix B*

1. Derive the centre of mass (CoM) position vectors in their respective frames.

2. Convert the derived position vectors to frame 0.

3. Calculate the position vectors from each joint's frame to all center of masses above it in the inertial frame.

4. Derive linear Jacobian matrices.

5. Derive angular Jacobian matrices.

6. Derive full Jacobian matrices.

Firstly, the CoM position vectors are derived using the diagram presented in section 2.1. The distance of each linkage's CoM to its respective frame is defined as shown below.

- l_{c_1}: Distance from linkage 1 CoM to frame 1.
- l_{c_2}: Distance from linkage 2 CoM to frame 2.
- l_{c_3}: Distance from linkage 3 CoM to frame 3.
- l_{c_4}: Distance from linkage 4 CoM to frame 4.

The CoM position vectors are then derived in their respective frames and converted to the inertial frame (frame 0).

$$^0P_{1c_1} = {}^0_1R \times \begin{bmatrix} 0 \\ 0 \\ l_{c_1} \end{bmatrix} \tag{9}$$

$$^0P_{2c_2} = {}^0_2R \times \begin{bmatrix} l_{c_2} \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

$$^0P_{3c_3} = {}^0_3R \times \begin{bmatrix} l_{c_3} \\ 0 \\ 0 \end{bmatrix} \tag{11}$$

$$^0P_{4c_4} = {}^0_4R \times \begin{bmatrix} 0 \\ -l_{c_4} \\ 0 \end{bmatrix} \tag{12}$$

The relative displacement between each joint's frame and all center of masses above it are calculated. Firstly, all the displacement vectors from joint 1 to all CoM's above it are evaluated.

$$^0P_{1c_2} = {}^0P_{12} + {}^0P_{2c_2} \tag{13}$$

$$^0P_{1c_3} = {}^0P_{13} + {}^0P_{3c_3} \tag{14}$$

$$^0P_{1c_4} = {}^0P_{14} + {}^0P_{4c_4} \tag{15}$$

Displacement vectors from joint 2 to all above CoMs.

$$^0P_{2c_3} = {}^0P_{23} + {}^0P_{3c_3} \tag{16}$$

$$^0P_{2c_4} = {}^0P_{24} + {}^0P_{4c_4} \tag{17}$$

Displacement vectors from joint 3 to all above CoMs.

$$^0P_{3c4} = {}^0P_{34} + {}^0P_{4c_4} \tag{18}$$

With all the necessary position vectors calculated, both the linear and angular Jacobians of CoM can be derived.

Firstly, the linear Jacobians $J_{vc_1}, J_{vc_2}, J_{vc_3}$ and J_{vc_4} of CoM are derived as shown below.

$$J_{vc_1} = \begin{bmatrix} {}^0\hat{z}_1 \times {}^0p_{1c_1} & (0,0,0)^T & (0,0,0)^T & (0,0,0)^T \end{bmatrix} \tag{19}$$

$$J_{vc_2} = \begin{bmatrix} {}^0\hat{z}_1 \times {}^0p_{1c_2} & {}^0\hat{z}_2 \times {}^0p_{2c_2} & (0,0,0)^T & (0,0,0)^T \end{bmatrix} \tag{20}$$

$$J_{vc_3} = \begin{bmatrix} {}^0\hat{z}_1 \times {}^0p_{1c_3} & {}^0\hat{z}_2 \times {}^0p_{2c_3} & {}^0\hat{z}_3 \times {}^0p_{3c_3} & (0,0,0)^T \end{bmatrix} \tag{21}$$

$$J_{vc_4} = \begin{bmatrix} {}^0\hat{z}_1 \times {}^0p_{1c_4} & {}^0\hat{z}_2 \times {}^0p_{2c_4} & {}^0\hat{z}_3 \times {}^0p_{3c_4} & {}^0\hat{z}_4 \times {}^0p_{4c_4} \end{bmatrix} \tag{22}$$

The angular Jacobians ($J_{\omega c_1}, J_{\omega c_2}, J_{\omega c_3}$ and $J_{\omega c_4}$) are derived and shown below.

$$J_{\omega c_1} = \begin{bmatrix} {}^0\hat{z}_1 & (0,0,0)^T & (0,0,0)^T & (0,0,0)^T \end{bmatrix} \tag{23}$$

$$J_{\omega c_3} = \begin{bmatrix} {}^0\hat{z}_1 & {}^0\hat{z}_2 & (0,0,0)^T & (0,0,0)^T \end{bmatrix} \tag{24}$$

$$J_{\omega c_1} = \begin{bmatrix} {}^0\hat{z}_1 & {}^0\hat{z}_2 & {}^0\hat{z}_3 & (0,0,0)^T \end{bmatrix} \tag{25}$$

$$J_{\omega c_1} = \begin{bmatrix} {}^0\hat{z}_1 & {}^0\hat{z}_2 & {}^0\hat{z}_3 & {}^0\hat{z}_4 \end{bmatrix} \tag{26}$$

The full Jacobian matrices of each CoM is as follows:

$$J_{c_1} = \begin{bmatrix} J_{vc_1} \\ J_{\omega c_1} \end{bmatrix} \tag{27}$$

$$J_{c_2} = \begin{bmatrix} J_{vc_2} \\ J_{\omega c_2} \end{bmatrix} \tag{28}$$

$$J_{c_3} = \begin{bmatrix} J_{vc_3} \\ J_{\omega c_3} \end{bmatrix} \tag{29}$$

$$J_{c_4} = \begin{bmatrix} J_{vc_4} \\ J_{\omega c_4} \end{bmatrix} \tag{30}$$

3.2 Derivation of Gravity Matrix

The gravity matrix is the expression of the static load of the robot due to the gravity forces acting on the CoMs of the robot. These gravity forces acting on the arm will need to be transformed into corresponding joint torques required to hold up the mass. The expression is shown below.

$$G(q) = \sum_{i=1}^{N} J_{c_i}^T (-m_i g^0 \hat{z}_1) \tag{31}$$

The G matrix for this specific robotic arm is as follows, where m_{c_i} represents the centre of mass for a given joint i. The maximum torque can be found by substituting the numerical values for elements such as length and joint positions into equation 32.

$$G(q) = g \times (J_{c_1}^T \times \begin{bmatrix} 0 \\ 0 \\ m_{c1} \\ 0 \\ 0 \\ 0 \end{bmatrix} + J_{c_2}^T \times \begin{bmatrix} 0 \\ 0 \\ m_{c2} \\ 0 \\ 0 \\ 0 \end{bmatrix} + J_{c_3}^T \times \begin{bmatrix} 0 \\ 0 \\ m_{c3} \\ 0 \\ 0 \\ 0 \end{bmatrix} + J_{c_4}^T \times \begin{bmatrix} 0 \\ 0 \\ m_{c4} \\ 0 \\ 0 \\ 0 \end{bmatrix}) \tag{32}$$

The G matrix expression has not been included since it is incredibly complex and cannot fit reasonably onto the page. The exact expression is included in the MATLAB file and can be viewed there.

3.3 Mass and Centre of Mass Estimation

The mass values can be estimated using the *Mass properties tool* in Onshape (the CAD software used to design the structure of the robot). Once the material of each part is specified, the software estimates the total mass of the parts combined, as well as the centres of mass of the combined parts (see Figure 5). By isolating each linkage of the arm, the estimated mass and centre of mass of each link can be determined. These values are collated in Table 1.

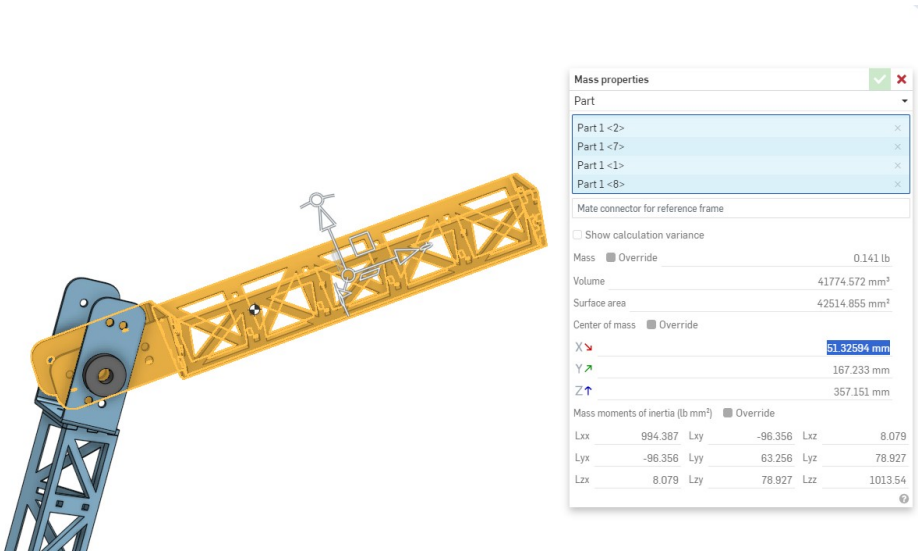


Figure 5: Onshape's Estimation of Linkage Mass and Centre of Mass.
Note: This figure purely acts as an example of how Onshape's estimation tool works. The mass properties values shown are to be ignored.

Section	Mass (kg)	CoM (m)
Link 1	0.26	0.03
Link 2	0.169	0.17
Link 3	0.1048	0.09
Link 4	0.098	0.06

Table 1: Summary of Linkage and Joint Masses and Centres of Mass. *Note that the centres of mass are represented in the respective linkage frame.*

The robot must also be able to pick up the heaviest chess piece, and so the mass of the payload is set to be the weight of the queen: $0.00569kg$. Finally, it is to be noted that in order to account for errors in the mass and centre of mass estimations, change in design, and dynamic loading of the robot: a safety factor of **1.3** is applied to all masses to ensure the assembled robot performs as intended.

3.4 Torque Requirement Analysis

Now that the wrench relationship and mechanical specifications are calculated and estimated: the magnitude of required joint torques to actuate the robot for a given configuration can be evaluated. Then, the required torque for the worst case configuration of the robot can be compared with the intended motor configuration to determine if the mechanical design is feasible. To do this, the required torque as calculated in Section 3.2 is calculated for configurations of the robot across the reachable workspace in a *MATLAB* for loop similar to that in Assignment 1. That is, the gravity vector, G (Equation 32), is evaluated using the masses and centres of masses given in Table 1 for a given set of joint angles determined using a nested for loop iterating through the expected ranges of each joint - thus producing the magnitude required joint torques across the reachable workspace. To demonstrate, the required torque on joint 2 across the robots reachable workspace is shown in Figure 6.

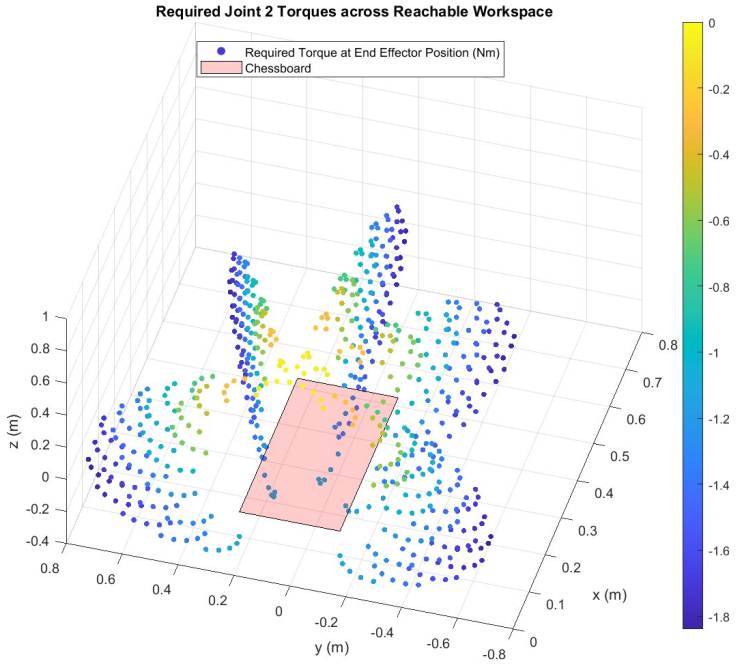


Figure 6: Required Torque for Joint 2 Across the Reachable Workspace

The maximum required torque for each joint is then the maximum calculated torque across the reachable workspace - which can be compared to the available torque at each joint from the servo motors. The summary of these values is shown in Table 2. Note that the available torque is calculated based on the servos' "rated continuous torque" which is equivalent to the nominal torque at operational speeds - taking into account that at lower speeds, the servos can produce more torque (up to $19.5 kg\ cm$), thus increasing the design margins.

Joint	Max. Required Torque (Nm)	Available Torque (Nm)	
		Nominal	*Peak*
1	0	0.64	1.91
2	1.84	1.28	3.82
3	0.66	0.64	1.91
4	0.084	0.2	0.59

Table 2: Summary of Calculated Maximum Required and Available Joint Torques

As can be seen from Table 2, joints 1 and 4 have more available nominal torque than is required, while joints 2 and 3 have less. This implies that under nominal operating speeds, joints 2 and 3 would fail to lift the weight of the robot and its payload under the worst case configuration. However, the required torque for those joints are well below the *peak* torques available, meaning that the joints theoretically could lift the robot at slower than nominal speeds. Since the required torque is only just above the nominal available torque but well below the peak available torque - it is assumed that the current joint configuration and mechanical design is still feasible, however further testing is required to verify that the motors will achieve the required torques at lower than nominal speeds.

4 Appendix A

$$ {}^{0}_{1}R = \begin{bmatrix} \cos(Q_1) & -\sin(Q_1) & 0 \\ \sin(Q_1) & \cos(Q_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{33} $$

$$ {}^{0}_{2}R = \begin{bmatrix} \cos(Q_1)\cos(Q_2) & -\cos(Q_1)\sin(Q_2) & \sin(Q_1) \\ \cos(Q_2)\sin(Q_1) & -\sin(Q_1)\sin(Q_2) & -\cos(Q_1) \\ \sin(Q_2) & \cos(Q_2) & 0 \end{bmatrix} \tag{34} $$

$$ {}^{0}_{3}R = \begin{bmatrix} \cos(Q_2+Q_3)\cos(Q_1) & -\sin(Q_2+Q_3)\cos(Q_1) & \sin(Q_1) \\ \cos(Q_2+Q_3)\sin(Q_1) & -\sin(Q_2+Q_3)\sin(Q_1) & -\cos(Q_1) \\ \sin(Q_2+Q_3) & \cos(Q_2+Q_3) & 0 \end{bmatrix} \tag{35} $$

$$ {}^{0}_{4}R = \begin{bmatrix} -\sigma_2 - \sigma_1 & -\sigma_4 - \sigma_3 & \sin(Q_1) \\ \sigma_4 - \sigma_3 & \sigma_1 - \sigma_2 & -\cos(Q_1) \\ \cos(Q_2+Q_3+Q_4) & -\sin(Q_2+Q_3+Q_4) & 0 \end{bmatrix} \tag{36} $$

where

$$ \sigma_1 = \frac{\sin(Q_2 - Q_1 + Q_3 + Q_4)}{2} $$

$$ \sigma_2 = \frac{\sin(Q_1 + Q_2 + Q_3 + Q_4)}{2} $$

$$ \sigma_3 = \frac{\cos(Q_2 - Q_1 + Q_3 + Q_4)}{2} $$

$$ \sigma_4 = \frac{\cos(Q_1 + Q_2 + Q_3 + Q_4)}{2} $$

$$ {}^{0}\hat{z}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{37} $$

$$ {}^{0}\hat{z}_2 = {}^{0}\hat{z}_3 = {}^{0}\hat{z}_4 = \begin{bmatrix} \sin(Q_1) \\ \cos(Q_1) \\ 0 \end{bmatrix} \tag{38} $$

5 Appendix B

$$ {}^{0}\mathbf{P}_{2c_2} = \begin{bmatrix} l_{c_2}\cos(Q_1)\cos(Q_2) \\ l_{c_2}\sin(Q_1)\cos(Q_2) \\ l_{c_2}\sin(Q_2) \end{bmatrix} \tag{39} $$

$$ {}^{0}\mathbf{P}_{3c_3} = \begin{bmatrix} l_{c_3}\cos(Q_2+Q_3)\cos(Q_1)\, l_{c_3}\cos(Q_2+Q_3)\sin(Q_1) \\ l_{c_3}\sin(Q_2+Q_3) \end{bmatrix} \tag{40} $$

$$ {}^{0}\mathbf{P}_{4c_4} = \begin{bmatrix} -l_{c_4}\left(\frac{\cos(Q_1+Q_2+Q_3+Q_4)}{2} + \frac{\cos(Q_2-Q_1+Q_3+Q_4)}{2}\right) \\ -\frac{l_{c_4}(\sin(Q_1+Q_2+Q_3+Q_4)-\sin(Q_2-Q_1+Q_3+Q_4))}{2} \\ -l_{c_4}\sin(Q_2+Q_3+Q_4) \end{bmatrix} \tag{41} $$

$$
{}^{0}\mathbf{P}_{1c_2} = \begin{bmatrix} l_{c_2}\cos\left(Q_1\right)\cos\left(Q_2\right) \\ l_{c_2}\cos\left(Q_2\right)\sin\left(Q_1\right) \\ l_{c_2}\sin\left(Q_2\right) \end{bmatrix} \tag{42}
$$

$$
{}^{0}\mathbf{P}_{1c_3} = \begin{bmatrix} \cos\left(Q_1\right)\left(l_{c_3}\cos\left(Q_2+Q_3\right)+l_2\cos\left(Q_2\right)\right) \\ \sin\left(Q_1\right)\left(l_{c_3}\cos\left(Q_2+Q_3\right)+l_2\cos\left(Q_2\right)\right) \\ l_{c_3}\sin\left(Q_2+Q_3\right)+l_2\sin\left(Q_2\right) \end{bmatrix} \tag{43}
$$

$$
{}^{0}\mathbf{P}_{1c_4} = \begin{bmatrix} \cos(Q_1)\,\sigma_1 - l_{c_4}\left(\frac{\cos(Q_1+Q_2+Q_3+Q_4)}{2}+\frac{\cos(Q_2-Q_1+Q_3+Q_4)}{2}\right) \\ \sin(Q_1)\,\sigma_1 - \frac{l_{c_4}(\sin(Q_1+Q_2+Q_3+Q_4)-\sin(Q_2-Q_1+Q_3+Q_4))}{2} \\ l_3\sin(Q_2+Q_3)+l_2\sin(Q_2)-l_{c_4}\sin(Q_2+Q_3+Q_4) \end{bmatrix} \tag{44}
$$

where

$$
\sigma_1 = l_3\cos(Q_2+Q_3)+l_2\cos(Q_2)
$$

$$
{}^{0}\mathbf{P}_{2c_3} = \begin{bmatrix} \cos(Q_1)\left(l_{c_3}\cos(Q_2+Q_3)+l_2\cos(Q_2)\right) \\ \sin(Q_1)\left(l_{c_3}\cos(Q_2+Q_3)+l_2\cos(Q_2)\right) \\ l_{c_3}\sin(Q_2+Q_3)+l_2\sin(Q_2) \end{bmatrix} \tag{45}
$$

$$
{}^{0}\mathbf{P}_{2c_4} = \begin{bmatrix} \cos(Q_1)\,\cos(Q_2)\,\sigma_1 - l_{c_4}\left(\frac{\cos(\sigma_5)}{2}+\frac{\cos(\sigma_4)}{2}\right)-\sigma_2 \\ \cos(Q_2)\,\sin(Q_1)\,\sigma_1 - \frac{l_{c_4}(\sin(\sigma_5)-\sin(\sigma_4))}{2}-\sigma_3 \\ l_3\,\sin(Q_2+Q_3)+l_2\,\sin(Q_2)-l_{c_4}\,\sin(Q_2+Q_3+Q_4) \end{bmatrix} \tag{46}
$$

where

$$
\sigma_1 = l_3\cos(Q_2+Q_3)+l_2\cos(Q_2)
$$

$$
\sigma_2 = l_3\cos(Q_1)\sin(Q_2)\sin(Q_3)
$$

$$
\sigma_3 = l_3\sin(Q_1)\sin(Q_2)\sin(Q_3)
$$

$$
\sigma_4 = Q_2 - Q_1 + Q_3 + Q_4
$$

$$
\sigma_5 = Q_1 + Q_2 + Q_3 + Q_4
$$

$$\mathbf{P}_{3c_4} = \begin{bmatrix} l_3 \cos(Q_2 + Q_3)\cos(Q_1) - l_{c_4}\left(\frac{\cos(\sigma_2)}{2} + \frac{\cos(\sigma_1)}{2}\right) \\ l_3 \cos(Q_2 + Q_3)\sin(Q_1) - l_{c_4}\left(\frac{\sin(\sigma_2)}{2} - \frac{\sin(\sigma_1)}{2}\right) \\ l_3 \sin(Q_2 + Q_3) - l_{c_4}\sin(Q_2 + Q_3 + Q_4) \end{bmatrix} \tag{47}$$

where

$$\sigma_1 = Q_2 - Q_1 + Q_3 + Q_4$$

$$\sigma_2 = Q_1 + Q_2 + Q_3 + Q_4$$